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COMMENT

Operator content of the four-state Potts quantum chain with $\mathbb{Z}_3$-boundary conditions

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Abstract. The four-state Potts quantum chain with a toroidal boundary condition leaving a global $\mathbb{Z}_3$ symmetry of the Hamiltonian is investigated. At the critical point, the infinite system shows SU(2) Kac-Moody symmetry. The conjectured operator content is confirmed by numerical finite-size calculations.

The four-state Potts quantum chain is defined by the self-dual Hamiltonian

$$H=-\frac{1}{\pi\sqrt{\lambda}}\sum_{j=1}^{N}\{(\sigma_j+\sigma_j^2+\sigma_j^3)+\lambda(\Gamma_j\Gamma_{j+1}^3+\Gamma_j^2\Gamma_{j+1}^2+\Gamma_j^3\Gamma_{j+1})\} \tag{1}$$

where σ and Γ are the 4×4 matrices

$$\sigma=\begin{pmatrix}1&0&0&0\\0&\omega&0&0\\0&0&\omega^2&0\\0&0&0&\omega^3\end{pmatrix}\qquad \Gamma=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix} \tag{2}$$

with $\omega=\exp(2\pi i/4)$. Here, N denotes the number of sites, λ plays the role of the inverse of temperature and the normalisation factor fixes the Euclidean timescale.

By investigation of the original Lagrangian formulation of the four-state Potts model (Potts 1952), Nienhuis and Knops (1985) found two spinor operators with spin $\frac{1}{3}$ and $\frac{2}{3}$ which have anomalous scaling dimensions

$$x_{1/3}=\Delta+\bar{\Delta}=\tfrac{4}{9}+\tfrac{1}{9}=\tfrac{5}{9}\qquad x_{2/3}=\Delta+\bar{\Delta}=\tfrac{25}{36}+\tfrac{1}{36}=\tfrac{13}{18}. \tag{3}$$

As will be seen in what follows, these operators can be obtained from the Hamiltonian (1) for certain toroidal boundary conditions defined below. It is the aim of this comment to give the full operator content for those boundary conditions, thus completing the knowledge of the operator content of this model with toroidal boundary conditions.

With periodic boundary conditions (BC), i.e. $\Gamma_{N+1}=\Gamma_1$, the Hamiltonian (1) shows a global S_4 symmetry according to the linear transformations

$$(\Gamma_j')^m=\sum_{n=1}^{3}A^{mn}(\Gamma_j)^n \tag{4}$$

where

$$A \in \Lambda = \{\Sigma^p C^q \Omega^r, 0 \leq p \leq 3, 0 \leq q \leq 1, 0 \leq r \leq 2\} \tag{5}$$

and the 3×3 matrices Σ, C and Ω are given by†

$$\Sigma = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Omega = \begin{pmatrix} -\frac{1}{2}i & \frac{1}{2}(1+i) & \frac{1}{2} \\ \frac{1}{2}(1+i) & 0 & \frac{1}{2}(1-i) \\ \frac{1}{2} & \frac{1}{2}(1-i) & \frac{1}{2}i \end{pmatrix}. \tag{6}$$

Here, Λ is a three-dimensional unitary irreducible representation of the symmetric group S_4. A torroidal BC 'B' is introduced by

$$(\Gamma_{N+1})^m = \sum_{n=1}^{3} B^{mn} (\Gamma_1)^n \tag{7}$$

where B is any of the 24 matrices of Λ (5). The global symmetry of the Hamiltonian (1) with BC B is in general smaller than S_4, in fact it is given by the subgroup $\mathrm{Cent}(B) \subset \Lambda$ (centraliser) formed by all matrices of Λ that commute with B, i.e.

$$\mathrm{Cent}(B) = \{A \in \Lambda \,|\, ABA^{-1} = B\}. \tag{8}$$

For conjugate elements of Λ these subgroups are conjugate subgroups of S_4 and therefore isomorphic. It follows that the global symmetry of (1) depends only on the conjugacy class of the element defining the boundary term, the same being obviously true for the finite-size spectrum. The five conjugacy classes of S_4 and the corresponding centralisers are given in table 1.

Table 1. Conjugacy classes and corresponding centralisers for the symmetric group S_4.

Conjugacy class	Cycle structure	Number of elements	Corresponding centraliser
$(\mathrm{I})_{S_4} = \{\mathbb{1}\}$	(1^4)	1	S_4
$(\mathrm{II})_{S_4} = \{\Sigma^2, \Sigma C, \Sigma^3 C\}$	(2^2)	3	D_4
$(\mathrm{III})_{S_4} = \{C, \Sigma^2 C, C\Omega, C\Omega^2, \Sigma\Omega, \Sigma^3\Omega^2\}$	$(1^2 2)$	6	$\mathbb{Z}_2 \otimes \mathbb{Z}_2$
$(\mathrm{IV})_{S_4} = \{\Sigma, \Sigma^3, \Sigma^3\Omega, \Sigma\Omega^2, \Sigma^2 C\Omega, \Sigma^2 C\Omega^2\}$	(4)	6	$\mathbb{Z}_4$
$(\mathrm{V})_{S_4} = \{\Omega, \Omega^2, \Sigma^2\Omega, \Sigma^2\Omega^2, \Sigma C\Omega, \Sigma C\Omega^2, \Sigma^3 C\Omega, \Sigma^3 C\Omega^2\}$	(13)	8	$\mathbb{Z}_3$

For all BC defined by elements of the conjugacy classes $(\mathrm{I})_{S_4}$, $(\mathrm{II})_{S_4}$, $(\mathrm{III})_{S_4}$ and $(\mathrm{IV})_{S_4}$, as well as for free BC, the operator content is known from previous studies on the Ashkin–Teller (AT) quantum chain (Baake *et al* 1987a, b, Yang 1987). The Hamiltonian of this model includes (1) for a special choice of the coupling constant. In general, the global symmetry group of the AT Hamiltonian is the dihedral group D_4, a subgroup of S_4 built by the eight matrices $\Sigma^p C^q$, $0 \leq p \leq 3$, $0 \leq q \leq 1$ (6). The five

† It should be noted that the similarity transformation corresponding to Ω does not conserve the algebra of observables but leaves the term $(\sigma_j + \sigma_j^2 + \sigma_j^3)$ in (1) invariant.

conjugacy classes of D_4 are given in table 2. Since the only conjugacy class of S_4 containing no element of D_4 is $(V)_{S_4}$, there is just one additional type of BC for the Hamiltonian (1), where it is left with a global $\mathbb{Z}_3$ symmetry. In what follows, we consider only the Hamiltonian (1) with BC Ω (6) and global $\mathbb{Z}_3$ symmetry $\mathrm{Cent}(\Omega) = \{\mathbb{1}, \Omega, \Omega^2\}$.

Table 2. Same as table 1, but for the dihedral group D_4.

Conjugacy class	Number of elements	Corresponding centraliser
$(\mathrm{I})_{D_4} = \{\mathbb{1}\}$	1	D_4
$(\mathrm{II})_{D_4} = \{\Sigma^2\}$	1	D_4
$(\mathrm{II})_{D_4} = \{\Sigma C, \Sigma^3 C\}$	2	$\mathbb{Z}_2 \otimes \mathbb{Z}_2$
$(\mathrm{IV})_{D_4} = \{C, \Sigma^2 C\}$	2	$\mathbb{Z}_2 \otimes \mathbb{Z}_2$
$(\mathrm{V})_{D_4} = \{\Sigma, \Sigma^3\}$	2	$\mathbb{Z}_4$

This remaining $\mathbb{Z}_3$ symmetry as well as translational invariance is used to prediagonalise the Hamiltonian. The eigenvalues split into three charge sectors labelled by $Q = 0, 1, 2$ corresponding to an eigenvalue $\exp(2\pi \mathrm{i} Q/3)$ of the matrix Ω. Let $E_k^Q(P, N)$ denote the eigenvalue of the N-site Hamiltonian belonging to the sector defined by charge Q and translational momentum P, where k counts the levels. $E_0(N)$ stands for the ground-state energy, i.e. the lowest eigenvalue of (1) with *periodic* BC. Consider the finite-size scaling limit of the spectrum given by the quantities (Cardy 1984, 1986a, b, von Gehlen and Rittenberg 1986)

$$\mathscr{E}_k^Q(P) = \lim_{N\to\infty} \frac{N}{2\pi} (E_k^Q(P, N) - E_0(N)). \tag{9}$$

It is already known (von Gehlen *et al* 1988, Baake *et al* 1987a, b) that, for theories with central charge $c = 1$ of the Virasoro algebra, the spectra (9) can be described in terms of unitary irreducible representations of two commuting U(1) Kac-Moody algebras. As a consequence of conformal invariance, the tensor product $(\Delta, \bar{\Delta})$ of two irreps gives the following contribution to the spectra (9):

$$\mathscr{E}_k^Q(P) = \Delta + r + \bar{\Delta} + \bar{r} \tag{10}$$

with a degeneracy obtained from the generating function $\Pi_V(z)\Pi_V(\bar{z})$, where

$$\Pi_V(z) = \prod_{m=1}^{\infty} (1 - z^m)^{-1}. \tag{11}$$

The (total) momentum p is given by

$$p = (\Delta + r) - (\bar{\Delta} + \bar{r}) = P + Q/3 \tag{12}$$

the spin s is defined by

$$s = \Delta - \bar{\Delta} \tag{13}$$

and the quantity

$$x = \Delta + \bar{\Delta} \tag{14}$$

is the scaling dimension of the operator.

In the scaling limit, the symmetry of the system examined goes beyond the U(1) Kac–Moody symmetry. It is described by the tensor product of two commuting shifted (or 'twisted') SU(2) Kac–Moody algebras (Goddard and Olive 1986, Baake *et al* 1988). Since in our case $c=1$, we have to consider only the two level-one representations which are characterised by $\varepsilon=0$ ($\frac{1}{2}$) for the isosinglet (isodoublet), respectively. Following the notation of Baake *et al* (1988) we denote a unitary irrep of the ρ-shifted SU(2) Kac–Moody algebra by $\langle\rho,\varepsilon\rangle$ where $\varepsilon=0,\frac{1}{2}$. The corresponding character expression is†

$$\begin{aligned}\chi_{\rho,\varepsilon}(z,y) &= \mathrm{Tr}(z^{L_0}y^{T_0^3}) \\ &= \sum_{m\in\mathbb{Z}} z^{(m+\varepsilon+\rho)^2} y^{m+\varepsilon+\rho}\Pi_V(z).\end{aligned} \tag{15}$$

Table 3. Finite-size scaling limits in the $Q=0$ sector for momenta $P=0, 1, 2, 3$. The numerical data (with estimated errors) are compared with the spectra obtained from the conjectured operator content.

P	$\Delta+r+\bar\Delta+\bar r$	$(\frac{1}{36},\frac{1}{36})$	$(\frac{1}{9},\frac{1}{9})$	$(\frac{4}{9},\frac{4}{9})$	$(\frac{25}{36},\frac{25}{36})$	$(\frac{49}{36},\frac{49}{36})$	$(\frac{16}{9},\frac{16}{9})$	$\mathscr{E}^0(P)$ (exp)
0	$\frac{1}{18}\cong 0.0556$	1	—	—	—	—	—	0.058 (3)
	$\frac{2}{9}\cong 0.222$	—	1	—	—	—	—	0.23 (1)
	$\frac{8}{9}\cong 0.889$	—	—	1	—	—	—	0.95 (6)
	$\frac{25}{18}\cong 1.389$	—	—	—	1	—	—	1.48 (6)
	$\frac{1}{18}+2\cong 2.056$	1	—	—	—	—	—	2.12 (8)
	$\frac{2}{9}+2\cong 2.222$	—	1	—	—	—	—	2.25 (4)
	$\frac{49}{18}\cong 2.722$	—	—	—	—	1	—	3.0 (2)
	$\frac{8}{9}+2\cong 2.889$	—	—	1	—	—	—	3.0 (1)
	$\frac{25}{18}+2\cong 3.389$	—	—	—	1	—	—	3.49 (8)
	$\frac{32}{9}\cong 3.556$	—	—	—	—	—	1	3.75 (8)
	$\frac{1}{18}+4\cong 4.056$	4	—	—	—	—	—	3.98 (8) 4.04 (8) 4.06 (8) 4.18 (10)
	$\frac{2}{9}+4\cong 4.222$	—	4	—	—	—	—	4.35 (12) 4.18 (10)
1	$\frac{1}{18}+1\cong 1.0556$	1	—	—	—	—	—	1.07 (2)
	$\frac{2}{9}+1\cong 1.222$	—	1	—	—	—	—	1.24 (2)
	$\frac{8}{9}+1\cong 1.889$	—	—	1	—	—	—	2.00 (8)
	$\frac{25}{18}+1\cong 2.389$	—	—	—	1	—	—	2.49 (8)
	$\frac{1}{18}+3\cong 3.056$	2	—	—	—	—	—	3.04 (3) 3.12 (8)
	$\frac{2}{9}+3\cong 3.222$	—	2	—	—	—	—	3.21 (5) 3.25 (5)
	$\frac{49}{18}+1\cong 3.722$	—	—	—	—	1	—	3.96 (10)
	$\frac{8}{9}+3\cong 3.889$	—	—	2	—	—	—	4.0 (1) 4.1 (2)
	$\frac{25}{18}+3\cong 4.389$	—	—	—	2	—	—	4.2 (2) 4.4 (2)
2	$\frac{1}{18}+2\cong 2.0556$	2	—	—	—	—	—	2.0555 (5) 2.07 (2)
	$\frac{2}{9}+2\cong 2.222$	—	2	—	—	—	—	2.21 (3) 2.25 (3)
	$\frac{8}{9}+2\cong 2.889$	—	—	2	—	—	—	3.01 (10) 3.05 (15)
	$\frac{25}{18}+2\cong 3.389$	—	—	—	2	—	—	3.47 (7) 3.49 (10)
	$\frac{1}{18}+3\cong 4.056$	4	—	—	—	—	—	4.00 (8) 4.02 (5) 4.06 (8) 4.16 (15)
	$\frac{2}{9}+4\cong 4.222$	—	4	—	—	—	—	4.2 (1) 4.2 (1) 4.3 (1) 4.3 (2)
3	$\frac{1}{18}+3\cong 3.0556$	3	—	—	—	—	—	3.05 (2) 3.05 (2) 3.07 (2)
	$\frac{2}{9}+3\cong 3.222$	—	3	—	—	—	—	3.20 (3) 3.21 (5) 3.24 (3)
	$\frac{8}{9}+3\cong 3.889$	—	—	3	—	—	—	3.93 (6) 4.0 (2) 4.05 (15)
	$\frac{25}{18}+3\cong 4.389$	—	—	—	3	—	—	4.34 (8) 4.47 (8)

† Note the identity $\chi_{\rho,\varepsilon}(z,y)=\chi_{\rho+\varepsilon,0}(z,y)$ which means that all representations alternatively can be regarded as shifts of the vacuum representation $\langle 0,0\rangle$.

Therefore, it can be decomposed into unitary U(1) Kac–Moody irreps (Δ) as follows:

$$\langle \rho, \varepsilon \rangle = \bigoplus_{m \in \mathbb{Z}} ((m + \varepsilon + \rho)^2). \tag{16}$$

Now, the conjecture for the operator content, previously proposed by Rittenberg (1987), shall be presented. Let $(\langle \rho, \varepsilon\rangle, \langle \bar{\rho}, \bar{\varepsilon}\rangle)$ denote the tensor product of two irreps of commuting shifted SU(2) Kac–Moody algebras. In this case, we need $\rho = \frac{1}{6}$ and hence

$$\begin{aligned} \langle \tfrac{1}{6}, 0\rangle &= \bigoplus_{m \in \mathbb{Z}} ((m + \tfrac{1}{6})^2) = \{\tfrac{1}{36}\} \oplus \{\tfrac{25}{36}\} \oplus \{\tfrac{49}{36}\} \\ \langle \tfrac{1}{6}, \tfrac{1}{2}\rangle &= \bigoplus_{m \in \mathbb{Z}} ((m + \tfrac{1}{3})^2) = \{\tfrac{1}{9}\} \oplus \{\tfrac{4}{9}\} \oplus \{\tfrac{16}{9}\} \end{aligned} \tag{17}$$

where

$$\begin{aligned} \{\tfrac{1}{36}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{36}(1 + 18k)^2) & \{\tfrac{1}{9}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{9}(1 + 9k)^2) \\ \{\tfrac{25}{36}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{36}(5 + 18k)^2) & \{\tfrac{4}{9}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{9}(2 + 9k)^2) \\ \{\tfrac{49}{36}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{36}(7 + 18k)^2) & \{\tfrac{16}{9}\} &= \bigoplus_{k \in \mathbb{Z}} (\tfrac{1}{9}(4 + 9k)^2). \end{aligned} \tag{18}$$

Table 4. Same as table 3 in the $Q = 1$ sector.

P	$\Delta + r + \bar{\Delta} + \bar{r}$	$(\frac{4}{9}, \frac{1}{9})$	$(\frac{1}{36}, \frac{25}{36})$	$(\frac{49}{36}, \frac{1}{36})$	$(\frac{16}{9}, \frac{4}{9})$	$(\frac{1}{9}, \frac{16}{9})$	$(\frac{25}{36}, \frac{49}{36})$	$(\frac{25}{9}, \frac{4}{9})$	$(\frac{121}{36}, \frac{1}{36})$	$\mathscr{E}^1(P)$ (exp)
0	$\frac{5}{9} \simeq 0.556$	1	—	—	—	—	—	—	—	0.54 (2)
	$\frac{26}{36} + 1 \simeq 1.722$	—	1	—	—	—	—	—	—	1.66 (8)
	$\frac{50}{36} + 1 \simeq 2.389$	—	—	1	—	—	—	—	—	2.49 (8)
	$\frac{5}{9} + 2 \simeq 2.556$	1	—	—	—	—	—	—	—	2.50 (6)
	$\frac{74}{36} + 1 \simeq 3.056$	—	—	—	—	—	1	—	—	2.9 (1)
	$\frac{20}{9} + 1 \simeq 3.222$	—	—	—	1	—	—	—	—	3.1 (1)
	$\frac{26}{36} + 3 \simeq 3.722$	—	2	—	—	—	—	—	—	3.6 (2) 3.71 (5)
	$\frac{17}{9} + 2 \simeq 3.889$	—	—	—	—	2	—	—	—	3.85 (9) 3.96 (5)
	$\frac{5}{9} + 4 \simeq 4.556$	4	—	—	—	—	—	—	—	4.3 (2) 4.4 (2)
1	$\frac{50}{36} \simeq 1.389$	—	—	1	—	—	—	—	—	1.43 (3)
	$\frac{5}{9} + 1 \simeq 1.556$	1	—	—	—	—	—	—	—	1.51 (3)
	$\frac{20}{9} \simeq 2.222$	—	—	—	1	—	—	—	—	2.05 (8)
	$\frac{26}{36} + 2 \simeq 2.722$	—	2	—	—	—	—	—	—	2.65 (8) 2.71 (5)
	$\frac{50}{36} + 2 \simeq 3.389$	—	—	1	—	—	—	—	—	3.46 (8)
	$\frac{5}{9} + 3 \simeq 3.556$	2	—	—	—	—	—	—	—	3.48 (8) 3.50 (9)
	$\frac{74}{36} + 2 \simeq 4.056$	—	—	—	—	—	2	—	—	3.82 (8) 3.8 (1)
	$\frac{20}{9} + 2 \simeq 4.222$	—	—	—	1	—	—	1	—	4.0 (2)
	$\frac{26}{36} + 4 \simeq 4.722$	—	3	—	—	—	—	—	—	4.5 (2)
2	$\frac{50}{36} + 1 \simeq 2.389$	—	—	1	—	—	—	—	—	2.44 (5)
	$\frac{5}{9} + 2 \simeq 2.556$	2	—	—	—	—	—	—	—	2.50 (5) 2.51 (4)
	$\frac{20}{9} + 1 \simeq 3.222$	—	—	—	1	—	—	1	—	3.05 (6) 3.47 (12)
	$\frac{26}{36} + 3 \simeq 3.722$	—	3	—	—	—	—	—	—	3.71 (5) 3.64 (6) 3.6 (2)
	$\frac{50}{36} + 3 \simeq 4.389$	—	—	2	—	—	—	—	1	4.24 (10) 4.4 (1) 4.1 (2)
	$\frac{5}{9} + 4 \simeq 4.556$	3	—	—	—	—	—	—	—	4.5 (2)
3	$\frac{50}{36} + 2 \simeq 3.389$	—	—	2	—	—	—	—	1	3.35 (6) 3.35 (9) 3.40 (6)
	$\frac{5}{9} + 3 \simeq 3.556$	3	—	—	—	—	—	—	—	3.43 (6) 3.48 (6) 3.52 (8)
	$\frac{20}{9} + 2 \simeq 4.222$	—	—	—	2	—	—	1	—	4.0 (1) 4.1 (1) 4.2 (2)

Table 5. Same as table 3 in the $Q=2$ sector.

P	$\Delta+r+\bar{\Delta}+\bar{r}$	$(\frac{25}{36},\frac{1}{36})$	$(\frac{1}{9},\frac{4}{9})$	$(\frac{49}{36},\frac{25}{36})$	$(\frac{16}{9},\frac{1}{9})$	$(\frac{1}{36},\frac{49}{36})$	$(\frac{4}{9},\frac{16}{9})$	$(\frac{25}{9},\frac{1}{9})$	$(\frac{121}{36},\frac{25}{36})$	$\mathscr{E}^2(P)$ (exp)
0	$\frac{26}{36}\simeq 0.722$	1	—	—	—	—	—	—	—	0.70 (2)
	$\frac{5}{9}+1\simeq 1.556$	—	1	—	—	—	—	—	—	1.49 (5)
	$\frac{74}{36}\simeq 2.056$	—	—	1	—	—	—	—	—	1.87 (5)
	$\frac{26}{36}+2\simeq 2.722$	1	—	—	—	—	—	—	—	2.67 (8)
	$\frac{17}{9}+1\simeq 2.889$	—	—	—	1	—	—	—	—	2.96 (8)
	$\frac{50}{36}+2\simeq 3.389$	—	—	—	—	2	—	—	—	3.37 (5) 3.47 (8)
	$\frac{5}{9}+3\simeq 3.556$	—	2	—	—	—	—	—	—	3.51 (9) 3.60 (9)
	$\frac{74}{36}+2\simeq 4.056$	—	—	1	—	—	—	—	—	3.82 (10)
	$\frac{20}{9}+2\simeq 4.222$	—	—	—	—	—	2	—	—	3.9 (2) 4.0 (2)
	$\frac{26}{36}+4\simeq 4.722$	4	—	—	—	—	—	—	—	4.5 (3)
1	$\frac{26}{36}+1\simeq 1.722$	1	—	—	—	—	—	—	—	1.70 (2)
	$\frac{17}{9}\simeq 1.889$	—	—	—	1	—	—	—	—	1.95 (4)
	$\frac{5}{9}+2\simeq 2.556$	—	2	—	—	—	—	—	—	2.49 (5) 2.58 (5)
	$\frac{74}{36}+1\simeq 3.056$	—	—	1	—	—	—	—	—	2.83 (7)
	$\frac{26}{36}+3\simeq 3.722$	2	—	—	—	—	—	—	—	3.61 (8) 3.66 (8)
	$\frac{17}{9}+2\simeq 3.889$	—	—	—	1	—	—	1	—	3.76 (8) 3.95 (9)
	$\frac{50}{36}+3\simeq 4.389$	—	—	—	—	3	—	—	—	4.34 (5) 4.3 (1) 4.3 (1)
	$\frac{5}{9}+4\simeq 4.556$	—	3	—	—	—	—	—	—	4.5 (2) 4.6 (2) 4.7 (2)
	$\frac{74}{36}+3\simeq 5.056$	—	—	2	—	—	—	—	—	4.9 (2)
2	$\frac{26}{36}+2\simeq 2.722$	2	—	—	—	—	—	—	—	2.70 (2) 2.70 (2)
	$\frac{17}{9}+1\simeq 2.889$	—	—	—	1	—	—	1	—	2.79 (4) 2.96 (3)
	$\frac{5}{9}+3\simeq 3.556$	—	3	—	—	—	—	—	—	3.44 (8) 3.50 (8) 3.58 (8)
	$\frac{74}{36}+2\simeq 4.056$	—	—	2	—	—	—	—	1	3.79 (8) 3.84 (8) 4.22 (9)
	$\frac{26}{36}+4\simeq 4.722$	3	—	—	—	—	—	—	—	4.6 (1) 4.7 (1)
3	$\frac{26}{36}+3\simeq 3.722$	3	—	—	—	—	—	—	—	3.56 (8) 3.68 (5) 3.5 (3)
	$\frac{17}{9}+2\simeq 3.889$	—	—	—	2	—	—	1	—	3.88 (7) 3.77 (8) 4.1 (3)

The operator content for the three charge sectors is then given by

$$\begin{aligned}
\mathscr{E}^0 &= (\{\tfrac{1}{36}\},\{\tfrac{1}{36}\})\oplus(\{\tfrac{25}{36}\},\{\tfrac{25}{36}\})\oplus(\{\tfrac{49}{36}\},\{\tfrac{49}{36}\})\oplus(\{\tfrac{1}{9}\},\{\tfrac{1}{9}\})\oplus(\{\tfrac{4}{9}\},\{\tfrac{4}{9}\})\oplus(\{\tfrac{16}{9}\},\{\tfrac{16}{9}\})\\
\mathscr{E}^1 &= (\{\tfrac{1}{36}\},\{\tfrac{25}{36}\})\oplus(\{\tfrac{49}{36}\},\{\tfrac{1}{36}\})\oplus(\{\tfrac{25}{36}\},\{\tfrac{49}{36}\})\oplus(\{\tfrac{4}{9}\},\{\tfrac{1}{9}\})\oplus(\{\tfrac{1}{9}\},\{\tfrac{16}{9}\})\oplus(\{\tfrac{16}{9}\},\{\tfrac{4}{9}\})\\
\mathscr{E}^2 &= (\{\tfrac{25}{36}\},\{\tfrac{1}{36}\})\oplus(\{\tfrac{1}{36}\},\{\tfrac{49}{36}\})\oplus(\{\tfrac{49}{36}\},\{\tfrac{25}{36}\})\oplus(\{\tfrac{1}{9}\},\{\tfrac{4}{9}\})\oplus(\{\tfrac{16}{9}\},\{\tfrac{1}{9}\})\oplus(\{\tfrac{4}{9}\},\{\tfrac{16}{9}\}).
\end{aligned} \tag{19}$$

The two spinor exponents obtained by Nienhuis and Knops (1985) (3) are contained in (19). Summing up the three sectors one obtains

$$\mathscr{E}^0\oplus\mathscr{E}^1\oplus\mathscr{E}^2 = (\langle\tfrac{1}{6},0\rangle,\langle\tfrac{1}{6},0\rangle)\oplus(\langle\tfrac{1}{6},\tfrac{1}{2}\rangle,\langle\tfrac{1}{6},\tfrac{1}{2}\rangle). \tag{20}$$

Note that the operator content in the charge sector Q is just given by those operators of (20) with spin $Q/3$ (up to an integer).

Finally, this conjecture has to be compared with the numerical data. These were obtained from numerical calculation of the energies $E_k^Q(P, N)$ for up to eight sites, applying Lanczos's algorithm (Lanczos 1950). Due to CP invariance, only positive momenta had to be considered. The quantities (9) were extrapolated using an algorithm due to Bulirsch and Stoer (1964) (see also Henkel and Schütz (1988)). The numerical data with estimated errors and the spectra deduced from (19) are compared in tables 3-5. As also previously observed (von Gehlen *et al* 1985, von Gehlen and Rittenberg 1986), the convergence is rather poor. This can be explained by the occurrence of logarithmic corrections to finite-size scaling (Cardy 1986b) expected in the presence of a marginal operator. Nevertheless, the observed agreement is good enough to confirm the conjecture.

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